Models of Set Theory II - Winter 2015/2016

Prof. Dr. Peter Koepke, Regula Krapf Problem sheet 8

Problem 31 (6 points)**.** Work in a model $\mathbb{M} \models \mathsf{GBC}$.

(a) Prove that every ccc class forcing is pretame.

(b) Prove that $\mathbb{P} = \{p : \mathrm{dom}(p) \to 2 \mid \mathrm{dom}(p) \subseteq \mathrm{Ord} \text{ finite}\}$, ordered by reverse inclusion, adds a proper class of Cohen reals.

(c) Show that pretameness does not imply the preservation of GBC.

Problem 32 (16 points)**.** Let M be a countable transitive model of ZFC. The goal of this excercise is to use class forcing to construct a model of $\mathsf{ZFC} + \mathsf{GCH}$. Let $\beth_\alpha = 2^\alpha$ be the *beth function* defined by the recursion

$$\beth_0 = \aleph_0, \quad \beth_{\alpha+1} = 2^{\beth_\alpha}, \quad \beth_\gamma = \bigcup_{\alpha<\gamma} \beth_\alpha \text{ for } \alpha \in \mathrm{Lim}.$$

Let $\mathbb{P}_\alpha = \mathrm{Fn}(\beth_\alpha^+, \beth_{\alpha+1}, \beth_\alpha^+)$. and let $\mathbb{P}$ denote the *Easton product* of the $\mathbb{P}_\alpha$, i.e. conditions are functions p with $\mathrm{dom}(p) \subseteq \mathrm{Ord}$ and $p(\alpha) \in \mathbb{P}_\alpha$ for each $\alpha \in \mathrm{dom}(p)$ such that for each strongly inaccessible cardinal λ, $|\mathrm{dom}(p) \cap \lambda| < \lambda$. Let

$$\mathbb{P}^{<\alpha} = \{p \upharpoonright \alpha \mid p \in \mathbb{P}\}$$
$$\mathbb{P}^{\geq\alpha} = \{p \upharpoonright \mathrm{Ord} \setminus \alpha \mid p \in \mathbb{P}\}.$$

(a) Prove that $|\mathbb{P}^{<\alpha}| \leq \beth_{\alpha+1}$.

(b) Prove that if α is a successor ordinal or α is a limit such that $\beth_\alpha$ is regular then $|\mathbb{P}^{<\alpha}| \leq \beth_\alpha$. *Hint: Prove first the successor case.*

(c) Show that for every ordinal α, $\mathbb{P}^{\geq\alpha}$ is $\beth_\alpha^+$-closed and for α as in (b), $\mathbb{P}^{<\alpha}$ has the $\beth_\alpha^+$-cc. Conclude that $\mathbb{P}$ preserves the axioms of $\mathsf{GB} + \mathsf{AC}$ (and in particular ZFC).

(d) Suppose that α is a limit ordinal such that $\beth_\alpha$ is singular with $\mathrm{cf}(\beth_\alpha) = \rho$ and $\langle \alpha_i \mid i < \rho \rangle$ cofinal in $\beth_\alpha$ with $\beth_{\alpha_0} > \rho$. Prove that the class

$$\begin{aligned} D = &\{p \in \mathbb{P} \mid \exists \langle A_i^\gamma \mid i < \rho, \gamma < \beth_{\alpha_i} \rangle (A_i^\gamma \text{ maximal antichain in } \mathbb{P}^{<\alpha_i} \wedge \\ &\forall i < \rho \, \forall \gamma < \beth_{\alpha_i} \, \forall q \in A_i^\gamma \, \exists \beta (\langle q, p^{\geq\alpha_i} \rangle \Vdash^M_{\mathbb{P}^{<\alpha_i} \times \mathbb{P}^{\geq\alpha_i}} \dot{f}(\check{\gamma}) = \check{\beta}))\} \end{aligned}$$

is dense for every $\mathbb{P}$-name $\dot{f}$ for a function $\beth_\alpha \to \beth_\alpha^+$.

(e) Prove that for every ordinal α, $(\beth_\alpha^+)^M$ is a cardinal in $M[G]$, where G is $\langle M, \mathrm{Def}(M) \rangle$-generic for $\mathbb{P}$. *Hint: Consider the cases that α is a successor ordinal, α is limit and $\beth_\alpha$ is regular, and $\beth_\alpha$ is singular separately.*

(f) Conclude that in $M[G]$, for each ordinal α, $\text{card}(\beth_\alpha^M)^{M[G]} = \aleph_\alpha^{M[G]}$.

(g) Show that for each cardinal $\kappa \in M[G]$ which is either of the form $\aleph_{\alpha+1}^{M[G]} = (\beth_\alpha^+)^M$ or $\aleph_\alpha^{M[G]} = \beth_\alpha^M$ such that $\beth_\alpha^M$ is a regular limit ordinal, $M[G] \models 2^\kappa = \kappa^+$.

(h) Suppose that $\kappa = \aleph_\alpha^{M[G]} = \beth_\alpha^M$ is a singular limit ordinal. Prove that for each $\lambda < \kappa$, $M[G] \models \kappa^\lambda \leq \kappa^+$.

(i) Conclude that in $M[G] \models \mathsf{ZFC} + \mathsf{GCH}$.

Hint: for (d) repeat the argument in the proof of Lemma 4.8 in the lecture notes on class forcing ρ-many times, i.e. construct a descending sequence $\langle p_i \mid i < \rho \rangle$ by repeating the argument in Lemma 4.8 at successor steps and consider $\bigcup_{i<\rho} p_i$. For (g), count nice names for subsets of $\beth_\alpha$.

Please hand in your solutions on Monday, 11.01.2016 before the lecture.